KNOX GRAMMAR SCHOOL

STATE

DA VINCI DECATHLON 2022

CELEBRATING THE ACADEMIC GIFTS OF STUDENTS

IN YEARS 7 & 8

MATHEMATICS

TEAM NUMBER _______________

1	2	3	4	5	6	7	8	Total	Rank
/12	/8	/6	/16	/6	/7	/7	/10	/72	

QUESTION 1: MATCHSTICK PATTERNS (12 MARKS)

Alice has a large pile of matchsticks which she uses to create sequences of diagrams.

a) Alice first uses the matchsticks to create rows of squares. The first 3 diagrams are shown below

(3 MARKS)

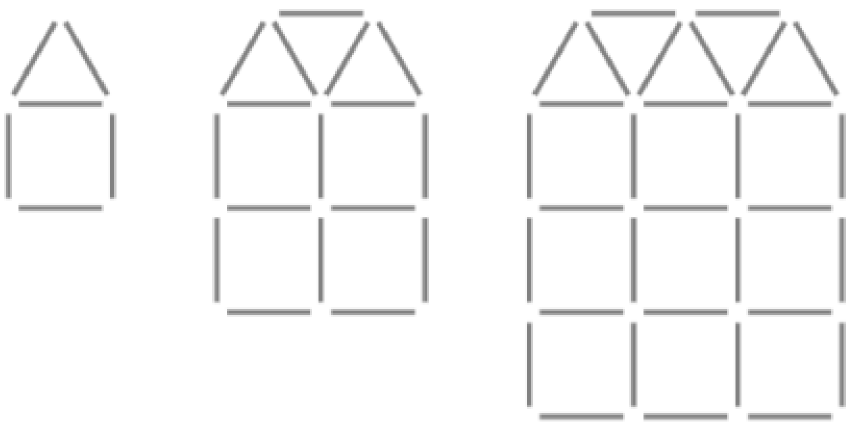
i. How many matchsticks will be needed to make the fourth diagram? __________

ii. How many matchsticks will be needed to make the tenth diagram? __________

iii. How many matchsticks will be needed to make the nth diagram (in terms of n)?

b) Alice gets a bit more creative and uses the matchsticks to create the following pattern:

(3 MARKS)



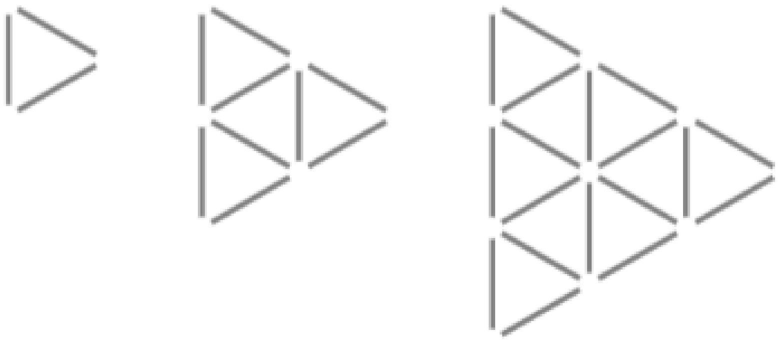
i. How many matchsticks will be needed to make the fourth diagram? __________

ii. How many matchsticks will be needed to make the tenth diagram? __________

iii. How many matchsticks will be needed to make the nth diagram (in terms of n)?

c) Alice decides she has enough with squares and creates the following pattern with only triangles

(6 MARKS)



i. How many matchsticks will be needed to make the first four diagrams?

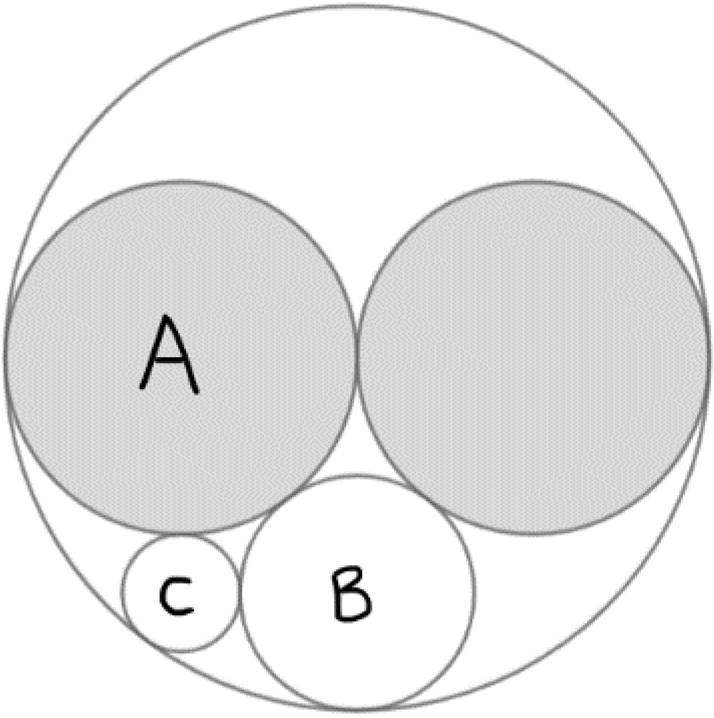
ii. How many matchsticks will be needed to make the first ten diagrams?

iii. How many matchsticks will be needed to make the first n diagrams (in terms of n)?

QUESTION 2: CIRCLES IN CIRCLES (8 MARKS)

The radius of the large circle is 1 and all circles are tangent to each other.

If the two shaded circles are congruent and touch at the center of the large circle, what are the radii of circles A, B and C?



QUESTION 3: FAST ADDITION (6 MARKS)

Mr Raft is a mathematics teacher who enjoys surprising his students. One day he picks two numbers and writes them on the board. Then he calculates the sum of the two most recent numbers written on the board and writes that sum onto the board. He continues until there are 10 numbers on the board.

a) What is a common name for the sequence he wrote?

(1 MARK)

b) Mr Raft claims he can calculate the sum of all the numbers on the board using the product of two numbers. Which two numbers does he multiply?

(5 MARKS)

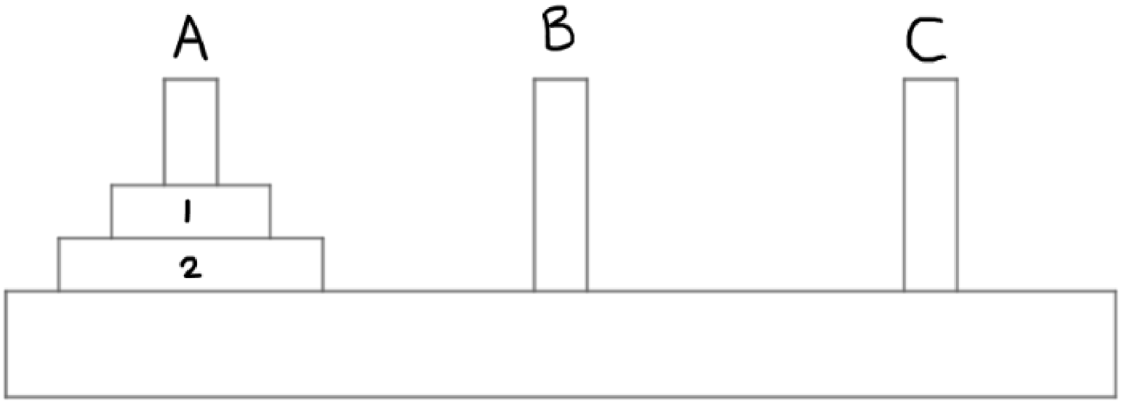
QUESTION 4: TOWER OF HANOI (16 MARKS)

The Tower of Hanoi is a mathematical puzzle which involves disks of various sizes and 3 rods. Initially, all the disks are placed on one rod by size with the largest at the bottom. The goal of the puzzle is to move the entire stack onto another rod while following the following three rules:

1. One disk can be moved at a time
2. Each move takes the disk at the top of one rod and moves it to the top of another rod.
3. A disk cannot be placed on top of a smaller disk.

a) Below is a Tower of Hanoi puzzle with 2 disks, write out the steps required to move the stack of disks to Pole C in 3 moves

(3 MARKS)



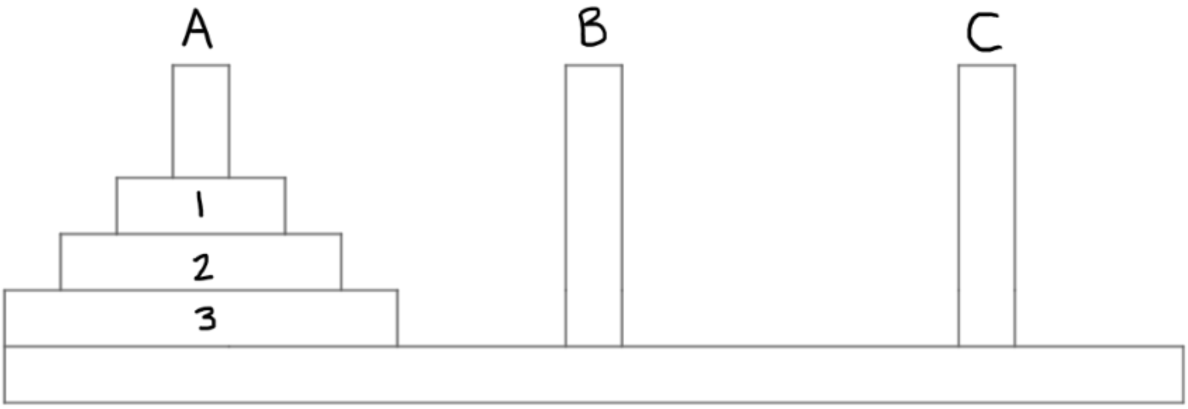
Move disk ______ from Pole ______ to Pole ______

Move disk ______ from Pole ______ to Pole ______

Move disk ______ from Pole ______ to Pole ______

b) Below is a Tower of Hanoi puzzle with 3 disks, write out the steps required to move the stack of disks to Pole C in the minimum number of moves.

(7 MARKS)



c) If T(n) represents the minimum number of moves required to solve the Tower of Hanoi puzzle with n disks, find and explain the relationship between T(n+1) and T(n)

(4 MARKS)

d) Find T(n) in terms of n and show that it satisfies the relationship found in c).

(2 MARKS)

QUESTION 5: NIM (6 MARKS)

In a game of Nim, there are two piles of coins on the table. One pile contains 26 coins while the other pile contains 30 coins. Alex and Bob take turns removing at least 1 coin from only 1 pile with Alex going first. The winner is the person who takes the last coin.

Who is the winner and what is their strategy?

QUESTION 6: ODD MAGIC SQUARES (7 MARKS)

A magic square is a square grid of numbers such that the numbers in each row, column and main diagonal add up to the same value. For example, the 3x3 magic square shown below has each row, column and diagonal add up to 15, while the 5x5 magic square shown below has each row, column and diagonal add up to 65.

3x3 magic square:

4	9	2
3	5	7
8	1	6

5x5 magic square:

11	18	25	2	9
10	12	19	21	3
4	6	13	20	22
23	5	7	14	16
17	24	1	8	15

Using the pattern in the magic squares above or otherwise, fill in the 7x7 magic square:
Hint 1: Look at one of the diagonals
Hint 2: Look at the placement of the 1

					19	

QUESTION 7: INDUCTION (7 MARKS)

Induction in mathematics is a technique usually used to prove a condition is true for infinite numbers.
Induction consists of two steps:

1. Show that the condition is true for the first number (Usually called base case)
2. Show that if the condition is true for n=k, then the condition is true for n=k+1

In this way, the condition is true for all integers after the first number.

a) Prove, using induction, that the sum of the first n odd numbers is equal to n^2 for all positive integers n. A template is given below.

Base case: n=1

(1 MARK)

Suppose the sum of the first k odd integers is k^2. Then, (show reasoning)

(2 MARKS)

b) Prove, using induction, that $4^{(3^r)} - 1$ is divisible 3^{r+1} for all positive integers r. You may use the fact that $4^n - 1$ is divisible by 3 for all integers n.

(4 MARKS)

QUESTION 8: TRAFFIC TROUBLES (10 MARKS)

When Daniel drives to work every day, he passes ten traffic lights, each either green, yellow or red. Daniel finds that due to the synchronization, a green light is always followed by a yellow light and a red light is never immediately followed by a red light.

a) How many possible sequences of ten lights are there?

(7 MARKS)

b) One day when Daniel was going to work, he noticed that the sixth light was yellow. How many possible sequences of ten lights are there for that day?

(3 MARKS)

END OF PAPER

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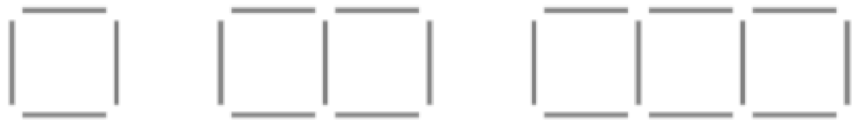
QUESTION 1: MATCHSTICK PATTERNS (12 MARKS)

Alice has a large pile of matchsticks which she uses to create sequences of diagrams.

a) Alice first uses the matchsticks to create rows of squares. The first 3 diagrams are shown below



i. How many matchsticks will be needed to make the fourth diagram? 13

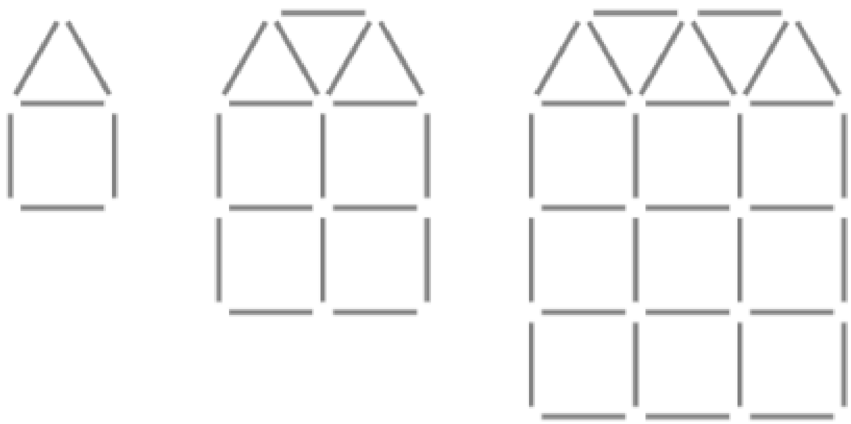
ii. How many matchsticks will be needed to make the tenth diagram? 31

iii. How many matchsticks will be needed to make the nth diagram (in terms of n)?

$3n + 1$ matchsticks

(1 mark for each correct answer)

b) Alice gets a bit more creative and uses the matchsticks to create the following pattern:



i. How many matchsticks will be needed to make the fourth diagram? 51

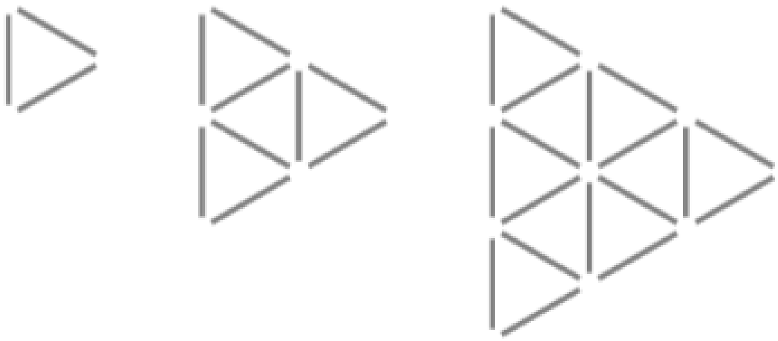
ii. How many matchsticks will be needed to make the tenth diagram? 249

iii. How many matchsticks will be needed to make the nth diagram (in terms of n)?

$2n^2 + 5n - 1$ matchsticks

(1 mark for each correct answer)

c) Alice decides she has enough with squares and creates the following pattern with only triangles



i. How many matchsticks will be needed to make the first four diagrams?

3 + 9 + 18 + 30 = 60 matchsticks

ii. How many matchsticks will be needed to make the first ten diagrams?

660 matchsticks

iii. How many matchsticks will be needed to make the first n diagrams (in terms of n)?

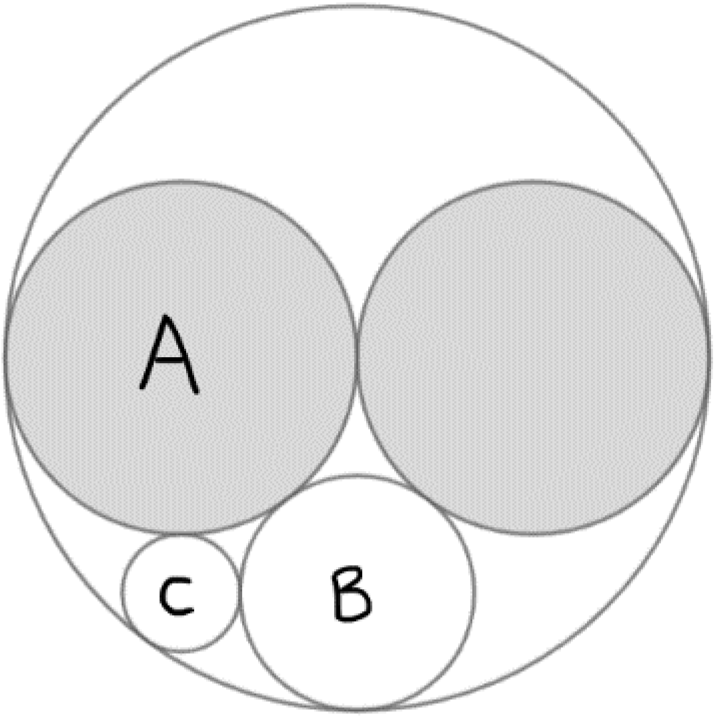
$$\frac{n(n+1)(2n+1)}{4} + \frac{3n(n+1)}{4} = \frac{2n^3 + 6n^2 + 4n}{4} = \frac{n^3 + 3n^2 + 2n}{2}$$

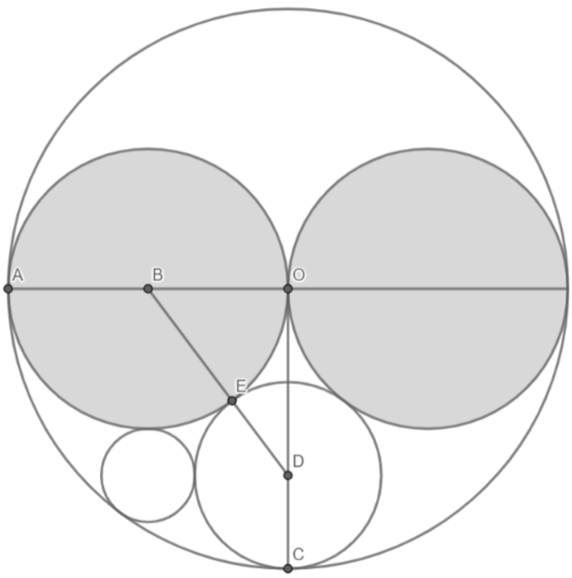
(2 marks for each correct answer)

QUESTION 2: CIRCLES IN CIRCLES (8 MARKS)

The radius of the large circle is 1 and all circles are tangent to each other.

If the two shaded circles are congruent and touch at the center of the large circle, what are the radii of circles A, B and C?





Let O be the center of the large circle.

Since the two shaded circles are congruent and touch at the center of the large circle, the radius of circle A is half the radius of the large circle

The radius of circle A if ½ (1 mark)

Label other points as shown in the diagram to the left.

Let r_B be the radius of circle B.

From symmetry (or otherwise) angle BOD is 90. (1 mark)

BO = ½ & OD = OC-CD = 1 - r_B & BD = BE + ED = ½ + r_B (1 mark)

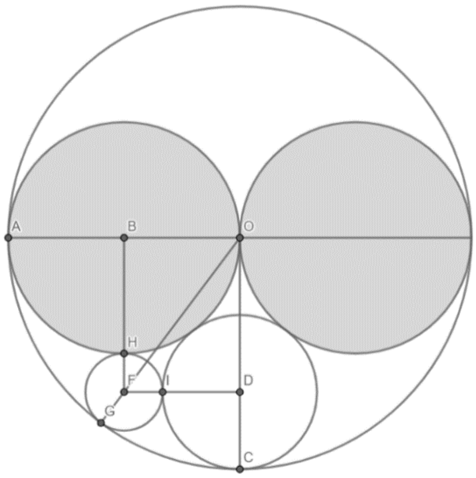
Since BOD is 90, from Pythagoras:

$$BO^2 + OD^2 = BD^2 \Rightarrow \left(\frac{1}{2}\right)^2 + (1 - r_B)^2 = \left(\frac{1}{2} + r_B\right)^2$$

$$\Rightarrow \frac{1}{4} + 1 - 2r_B + r_B^2 = \frac{1}{4} + r_B + r_B^2$$

$$\Rightarrow 1 = 3r_B \Rightarrow r_B = \frac{1}{3}$$

So the radius of circle B is 1/3 (1 mark)



Label the points as shown on the left where F is the point such that angle FBO and FDO are 90 degrees.

G is the intersection of the big circle and FO

H is the intersection of circle A and FB

I is the intersection of circle B and FD

Since FBO, FDO and BOD are 90 degrees, BODF is a rectangle. So:

FH = BF – BH = OD – ½ = OC – DC – ½ = 1 – 1/3 - ½ = 1/6 (1 mark)

FI = FD – ID = BO – ID = ½ - 1/3 = 1/6 (1 mark)

Since ODF is 90, we can use Pythagoras and get:

$FG = GO - OF = 1 - \sqrt{OD^2 + FD^2} = 1 - \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = 1 - \sqrt{\frac{25}{36}} = 1 - \frac{5}{6} = \frac{1}{6}$ (1 mark)

So the circle with center F and radius 1/6 is tangent to circle A, circle B and the big circle and hence is circle C.

So the radius of circle C is 1/6 (1 mark)

Note: If F is defined as the center of the circle C instead of the intersection of the line perpendicular to BO through B and the line perpendicular to DO through D. Give no marks for finding FH, FI and FG if they assume FBO and FDO are perpendicular without proof.

QUESTION 3: FAST ADDITION (6 MARKS)

Mr Raft is a mathematics teacher who enjoys surprising his students. One day he picks two numbers and writes them on the board. Then he calculates the sum of the two most recent numbers written on the board and writes that sum onto the board. He continues until there are 10 numbers on the board.

a) What is a common name for the sequence he wrote?

Fibonacci sequence (1 mark)

b) Mr Raft claims he can calculate the sum of all the numbers on the board using the product of two numbers. Which two numbers does he multiply?

If a is the first number and b is the second number, then the ten numbers on the board are:

N	1	2	3	4	5	6	7	8	9	10
N^{th} number	a	b	a + b	a + 2b	2a + 3b	3a + 5b	5a + 8b	8a + 13b	13a + 21b	21a + 34b

So, the sum of all 10 numbers is 55a + 88b

Notice that 55a + 88b = 11 * (5a + 8b)

And 5a + 8b corresponds to the 7^{th} number in the list

So, the two numbers he multiplies is 11 and the 7^{th} number in the list.

(2 marks for finding the 10 numbers in terms of the 1^{st} and 2^{nd} numbers)

(1 mark for finding the sum of all 10 numbers in terms of a and b (or whatever variables they used))

(1 mark for noticing the factor of 11)

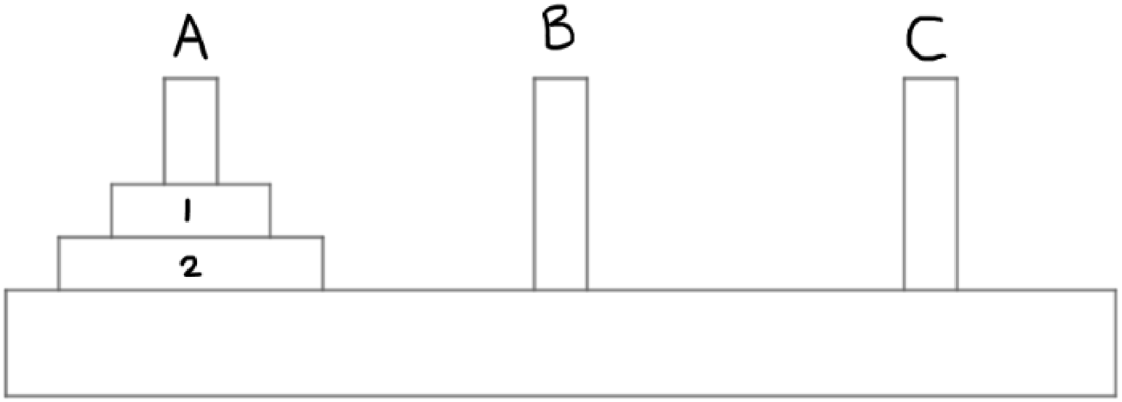
(1 mark for realizing the other factor is the 7^{th} number in the list)

QUESTION 4: TOWER OF HANOI (16 MARKS)

The Tower of Hanoi is a mathematical puzzle which involves disks of various sizes and 3 rods. Initially, all the disks are placed on one rod by size with the largest at the bottom. The goal of the puzzle is to move the entire stack onto another rod while following the following three rules:

1. One disk can be moved at a time
2. Each move takes the disk at the top of one rod and moves it to the top of another rod.
3. A disk cannot be placed on top of a smaller disk.

a) Below is a Tower of Hanoi puzzle with 2 disks, write out the steps required to move the stack of disks to Pole C in 3 moves



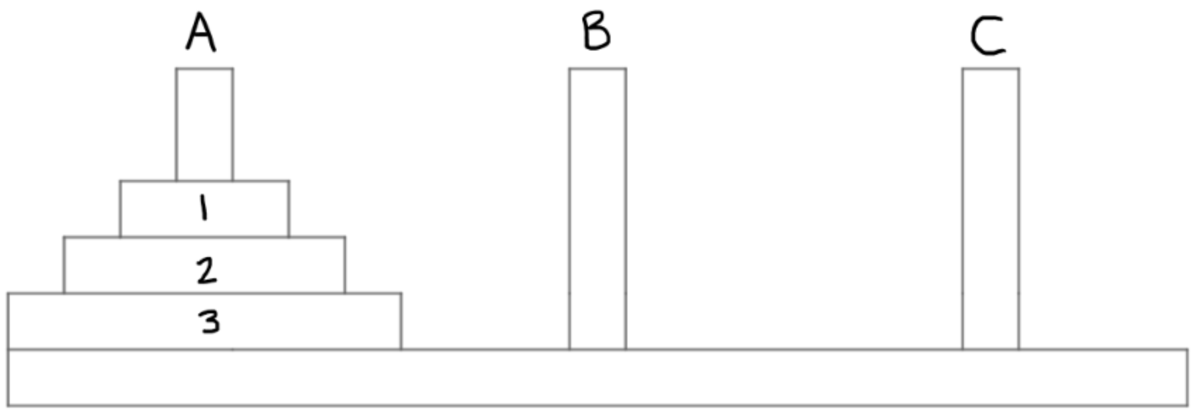
Move disk 1 from Pole A to Pole B

Move disk 2 from Pole A to Pole C

Move disk 1 from Pole B to Pole C

(1 Mark for each correct move)

b) Below is a Tower of Hanoi puzzle with 3 disks, write out the steps required to move the stack of disks to Pole C in the minimum number of moves.



Move disk 1 from Pole A to Pole C

Move disk 2 from Pole A to Pole B

Move disk 1 from Pole C to Pole B

Move disk 3 from Pole A to Pole C

Move disk 1 from Pole B to Pole A

Move disk 2 from Pole B to Pole C

Move disk 1 from Pole A to Pole C

(1 mark for each correct move)

Note: Give 4 marks for correction solution but going over 7 moves

c) If T(n) represents the minimum number of moves required to solve the Tower of Hanoi puzzle with n disks, find and explain the relationship between T(n+1) and T(n)

Since the disk n+1 would be the biggest disk and everything can go on top of it we can only move it if there's nothing on top of it and a rod is empty.

So the minimum number of moves requires the other n disks to be moved to the non-goal pole, moving disk n+1 to the goal pole, and then moving the other n disks to the goal pole. (1 mark)

Moving the n disks requires a minimum of T(n) moves while moving disk n+1 takes 1 move. (2 marks)

So the relationship is T(n+1) = 2*T(n) + 1 (1 mark)

d) Find T(n) in terms of n and show that it satisfies the relationship found in c).

$T(n) = 2^n - 1$ (1 mark)

And $T(n+1) = 2^{n+1} - 1 = (2^n - 1) + (2^n - 1) + 1 = 2 \times T(n) + 1$ (1 mark)

QUESTION 5: NIM (6 MARKS)

In a game of Nim, there are two piles of coins on the table. One pile contains 26 coins while the other pile contains 30 coins. Alex and Bob take turns removing at least 1 coin from only 1 pile with Alex going first. The winner is the person who takes the last coin.

Who is the winner and what is their strategy?

Alex is the winner (1 mark)

On their first move, Alex should take 4 coins from the pile with 30 coins (1 mark)

This makes the number of coins in each pile equal (1 mark)

From then on, Alex will take the same number of coins Bob takes but from the other pile (1 mark)

This ensures that after Alex's turn the number of coins in each pile remain the same, and after Bob's turn, the number of coins in each pile is different. (1 mark)

So since the person who takes the last coin leaves 2 equal piles (of 0 coins), Alex must be the winner (1 mark)

Note: A vague solution of 'making the piles even' without going into the details should be awarded 3 marks, +1 if they mention that Alex is the winner

QUESTION 6: ODD MAGIC SQUARES (7 MARKS)

A magic square is a square grid of numbers such that the numbers in each row, column and main diagonal add up to the same value. For example, the 3x3 magic square shown below has each row, column and diagonal add up to 15, while the 5x5 magic square shown below has each row, column and diagonal add up to 65.

3x3 magic square:

4	9	2
3	5	7
8	1	6

5x5 magic square:

11	18	25	2	9
10	12	19	21	3
4	6	13	20	22
23	5	7	14	16
17	24	1	8	15

Using the pattern in the magic squares above or otherwise, fill in the 7x7 magic square:
Hint 1: Look at one of the diagonals
Hint 2: Look at the placement of the 1

22	31	40	49	2	11	20
21	23	32	41	43	3	12
13	15	24	33	42	44	4
5	14	16	25	34	36	45
46	6	8	17	26	35	37
38	47	7	9	18	27	29
30	39	48	1	10	19	28

Give one mark each for putting all numbers in the following sets in the correct spots
{1, 2, 3, 4, 5, 6, 7}
{8, 9, 10, 11, 12, 13, 14}
{15, 16, 17, 18, 19, 20, 21}
{22, 23, 24, 25, 26, 27, 28}
{29, 30, 31, 32, 33, 34, 35}
{36, 37, 38, 39, 40, 41, 42}
{43, 44, 45, 46, 47, 48, 49}

(Notice each set runs on the same diagonal (with warping))

QUESTION 7: INDUCTION (7 MARKS)

Induction in mathematics is a technique usually used to prove a condition is true for infinite numbers.
Induction consists of two steps:

1. Show that the condition is true for the first number (Usually called base case)
2. Show that if the condition is true for n=k, then the condition is true for n=k+1

In this way, the condition is true for all integers after the first number.

a) Prove, using induction, that the sum of the first n odd numbers is equal to n^2 for all positive integers n. A template is given below.

Base case: n=1

$1 = 1^2$ (1 mark)

Base case true

Suppose the sum of the first k odd integers is k^2. Then

Since the k^{th} odd number is 2k-1 (1 mark)

The sum of the first k+1 odd integers is:

$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = (1 + 3 + 5 + \cdots + (2k-1)) + (2k+1)$
$= (k^2) + (2k+1)$ (1 mark – for substitution of (1+3+…+2k-1) to k^2)
$= (k+1)^2$

So the inductive step is true and the sum of the first n odd numbers is equal to n^2 for all positive integers n

b) Prove, using induction, that $4^{(3^r)} - 1$ is divisible 3^{r+1} for all positive integers r. You may use the fact that $4^n - 1$ is divisible by 3 for all integers n.

Base case r=0:

$$4^{3^0} - 1 = 4^1 - 1 = 4 - 1 = 3$$
$$3^{0+1} = 3$$

3 divides 3 and the base case is true

OR

Base case r=1:

$$4^{(3^1)} - 1 = 4^3 - 1 = 64 - 1 = 63$$
$$3^{1+1} = 9$$

9 divides 63 and the base case is true

Suppose 3^{k+1} divides $4^{3^k} - 1$ for some integer k

Consider $4^{3^{k+1}} - 1 = \left(4^{3^k}\right)^3 - 1$

By factorization for the difference of two perfect cubes we get

$$4^{3^{k+1}} - 1 = \left(4^{3^k} - 1\right)\left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right)$$

$$\left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right) = \left(4^{2\times 3^k} + 4^{3^k} + 1\right)$$
$$= \left(4^{2\times 3^k} - 1 + 4^{3^k} - 1 + 3\right)$$

Since $4^{2\times 3^k} - 1$, $4^{3^k} - 1$, and 3 are all divisible by 3,
The sum of the 3 is divisible by 3 and so
$\left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right)$ is divisible by 3

Since 3^{k+1} divides $4^{3^k} - 1$ for some integer k

Then $3^{k+2} = 3^{k+1} \times 3$ divides $\left(4^{3^k} - 1\right) \times \left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right) = 4^{3^{k+1}} - 1$
So the inductive step is true and $4^{(3^r)} - 1$ is divisible 3^{r+1} for all positive integers r.

(1 mark – Proving base case is true)
(1 mark – Factorizing $4^{3^{k+1}} - 1$ using difference of perfect cubes)
(1 mark – Proving $\left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right)$ is divisible by 3)
(1 mark – Using the fact that 3^{k+1} divides $4^{3^k} - 1$ for some integer k and finishing the proof)

Note: Another way to prove $\left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right)$ is divisible by 3 using modular arithmetic is shown below and should be awarded the 3rd mark

$$\left(\left(4^{3^k}\right)^2 + 4^{3^k} + 1\right) \equiv \left(\left(1^{3^k}\right)^2 + 1^{3^k} + 1\right) \equiv 1 + 1 + 1 \equiv 3 \equiv 0\ (mod\ 3)$$

QUESTION 8: TRAFFIC TROUBLES (10 MARKS)

When Daniel drives to work every day, he passes ten traffic lights, each either green, yellow or red. Daniel finds that due to the synchronization, a green light is always followed by a yellow light and a red light is never immediately followed by a red light.

a) How many possible sequences of ten lights are there?

7 MARKS

If a green light is always followed by a yellow light, and a red light is never immediately followed by a red light, then:

- Only a yellow light can go before a red light
- Any light can go before a yellow light
- Only a red or yellow light can go before a green light.

So if R(n) represents the number of sequences of n lights, where the last light is red and
If Y(n) represents the number of sequences of n lights, where the last light is yellow and
If G(n) represents the number of sequences of n lights, where the last light is green

Then
$R(n+1) = Y(n)$ (1 mark)
$Y(n+1) = R(n) + Y(n) + G(n)$ (1 mark)
$G(n+1) = R(n) + Y(n)$ (1 mark)

Since $R(1) = Y(1) = G(1) = 1$ (1 mark)
We can create the following table:

n	1	2	3	4	5	6	7	8	9	10
R(n)	1	1	3	6	13	28	60	129	277	595
Y(n)	1	3	6	13	28	60	129	277	595	1278
G(n)	1	2	4	9	19	41	88	189	406	872

(2 marks for table)
So the total number of possible sequences of ten lights would be $R(10) + Y(10) + G(10) = 2745$ (1 mark – for answer)

b) One day when Daniel was going to work, he noticed that the sixth light was yellow. How many possible sequences of ten lights are there for that day?

3 MARKS

From part a, there are 60 sequences for the first 6 lights if the 6th light is yellow. (1 mark)

Since the 6th light is yellow, we can use $R(6) = G(6) = 0$ and $Y(6) = 60$ to get the following table

n	6	7	8	9	10
R(n)	0	60	60	180	360
Y(n)	60	60	180	360	780
G(n)	0	60	120	240	540

(1 mark for table)
So the number of possible sequences would be 360+780+540 = 1680 (1 mark – for answer)

Note: Using $R(6) = G(6) = 0$ and $Y(6) = 1$ and then multiplying 6+13+9 by 60 is also acceptable and should be awarded the full marks for part b) (3 marks)

Note: Other correct ways of finding the answer with full working should be awarded full marks