KNOX GRAMMAR SCHOOL

STATE

DA VINCI DECATHLON 2021

CELEBRATING THE ACADEMIC GIFTS OF STUDENTS

IN YEARS 7 & 8

MATHEMATICS

TEAM NUMBER ______________

1	2	3	4	5	6	7	8	9	Total	Rank
/18	/5	/6	/18	/6	/3	/10	/4	/7	/80	

Chance is most commonly related to probability-based mathematics. More broadly, however, mathematics is often about removing the need to rely on random chance as an explanation by deducing or calculating possibilities. By finding patterns and connections or processes and methods it is possible to remove 'coincidence' or 'fate' and instead create rational justifications for peculiar results. This paper will explore various unusual findings and seemingly coincidental results. Your task will be to employ mathematical reasoning to remove these ambiguities of chance.

QUESTION 1: PARTITION FATE

18 MARKS

(a) Write all 14 ways that 7 can be written as the sum of positive integers (e.g. 6 + 1). We call each of these a ‘partition’. Order does not matter, so 6 + 1 is the same as 1 + 6 and only one is to be included in the 15. (3 marks)

3 marks for all 14 2 marks for more than 10 1 mark for more than 5	6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1+1

(b) How many partitions use only odd integers? (1 mark)

1 mark	5

(c) How many partitions have no repeated integers (“distinct”)? (1 mark)

1 mark	5

(d) By considering the questions in (b) and (c) for numbers less than 7, deduce a theory that connects the number of “odd” partitions to the number of “distinct” partitions. You need not consider all numbers less than 7, only enough for you to be satisfied of a theory. (2 marks)

1 mark for odd = distinct 1 mark for mathematical language (e.g. positive integer n)	*For any positive integer* n, *the number of “odd” partitions of* n *is the same as the number of “distinct” partitions of* n.

(e) Determine how *many “distinct” partitions exist for an integer that has 24 “odd” partitions.* (1 mark)

1 mark	*24 odd partitions = 24 distinct partitions.*

(f) It is possible to transform between “odd” and “distinct” partitions. Whenever two integer parts of a partition are the same, add them together and repeat until all are distinct. Below is an odd partition of 28. Using this technique, prove the odd partition transforms into the distinct partition “12+7+6+2+1” (2 marks)

7+3+3+3+3+3+3+1+1+1

1 mark	7+(3+3)+(3+3)+(3+3)+(1+1)+1
1 mark	→7+(6+6)+6+2+1 →12+7+6+2+1

(g) By reversing the distinct partition “12+7+6+2+1” to the odd partition, describe a general technique for transforming a “distinct” partition into an “odd” partition. (3 marks)

1 mark	Whenever there is an even part, cut it in half and write it twice.
1 mark	12+7+6+2+1 from above becomes (6+6)+7+(3+3)+(1+1)+1
1 mark	(3+3)+(3+3)+7+3+3+1+1+1

(h) Identify all the odd partitions of 5 and transform them into distinct partitions. (4 marks)

1 mark	3 + 1 + 1 and 1 + 1 + 1 + 1 identified
1 mark	3 + 1 + 1 → 3 + (1+1)
1 mark	1 + 1 + 1 + 1 + 1 → (2) + (2) + 1
1 mark	3+2 and 4+1 as answers respectively

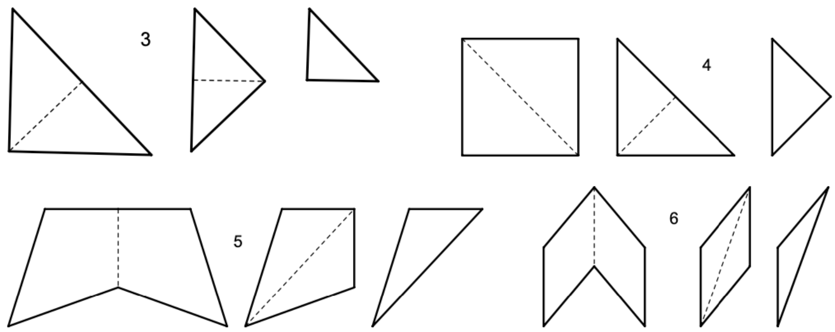
(i) How do the relationships in (e)-(g) support your theory in (d)? (1 mark)

1 mark	If an odd always has a corresponding distinct pair, then there must be an equal number of distinct pairs as odd pairs.

QUESTION 2: TRIANGULATE THE ORIGIN

5 MARKS

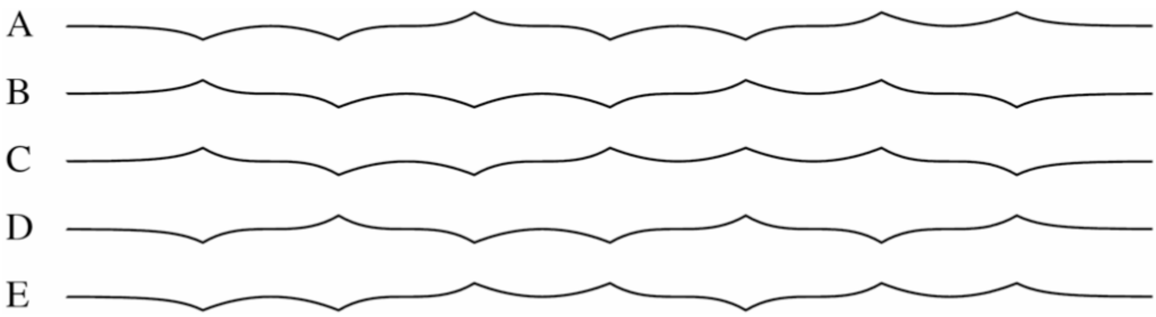
Joe is given a polygon shaped piece of paper. It is folded in half along a line of symmetry. The resulting shape is then also folded in half along a line of symmetry. This produces a triangle. How many different numbers of sides could the original polygon have had for this to occur? What are the possible numbers of sides?

1 mark	4 side lengths
3 marks (1 mark each)	3, 4, 5 and 6
	When unfolding the final triangle, the edge of this triangle that was the crease is in the interior of the previous polygon. The other two edges of the final triangle have mirror images in the previous polygon, which can be extensions of the sides of the triangle. The intermediate polygon can therefore have at most 4 sides. The original polygon can have at most 6 sides. The possibilities for the number of sides of the original polygon are therefore 3, 4, 5 and 6. See below: 

QUESTION 3: PAPER PROBLEMS

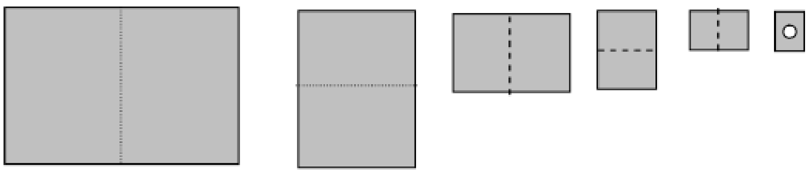
6 MARKS

(a) A detective reaches a crime scene. She had previously been given a photo that contained a strip of paper folded over in the middle three times that contained a secret invisible message. At the scene, however, the paper could not be located. Instead, the detective found 5 pieces of unfolded paper. The below is a view of these papers from the side. Which of the below cannot be the original folded paper? (2 marks)



1 mark	D
1 mark - explanation	E.g. If folding a strip is to result in one of the shapes in the figure, the left half must be the same as the right, only oriented in the opposite direction. This is true of all five strips shown. However, the same must hold in the left (or right) half only; the left half of each half must be the same as the right, albeit upside-down. This is not the case for D, and this strip is, therefore, not possible.

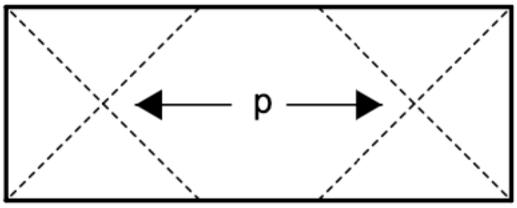
(b) The detective is in search for another clue page. This page had been folded five times as shown below, with a hole made through the final folded paper. The detective sees unfolded pages around an office, all with various numbers of holes. How many holes will there be in the unfolded paper the detective is in search for? (2 marks)



1 mark	32
1 mark - explanation	Each fold doubles the number of layers of paper. The final small rectangle is, therefore, made up of 32 layers = 32 holes.

(c) The third clue is a train ticket *a* cm long and *b* cm wide. The ticket has been creased as indicated below, where the creases bisect the angles in the corners.

To determine the correct ticket, the detective needs to know the value the distance *p* in the diagram below. Determine *p* in terms of *a* and *b*. (2 marks)



1 mark	*a*-*b*
1 mark - explanation	Since the intersection is at centre of a square of side *b* distance is *b*/2. p is therefore *a*-2(*b*/2) = *a*-*b*.

QUESTION 4: CHANCE ENCOUNTERS

18 MARKS

Below are a series of applications of well-known chance based mathematical results. Read the background information and answer all questions with mathematical reasoning; very few marks will be rewarded simply for the result.

You will need to know two rules of probability:

Where events are dependent on each other, e.g. the grass is green **AND** there is a butterfly, the probability = P(grass is green) x P(there is a butterfly). An 'and' usually indicates these types of events.

Where events are independent, you add the probabilities of each event. E.g. P(grass is green OR there is a butterfly) = P(grass is green) + P(there is a butterfly).

(a) Mona attends a decathlon with 24 other people.

a. Assuming there are 365 days in a year, explain why the probability that two people don't have the same birthday is 365/365 x 364/365 = 99.72% (1 mark)

1 mark	You select one person, who can have any birthday. The second person must not have that particular day, leaving 364 days left so the P is 364/365 for the second and you multiply both as the events are dependent.

b. What is the probability that three people don't have the same birthday? (2 marks)

1 mark	Answer above x 363/365
1 mark	= 99.17%

c. By determining a pattern from the above results, what is the probability that all 25 people don't have the same birthday? (3 marks)

1 mark	P=365/365x365/365x…….x341/365
1 mark	= 365x364x…..x341/$(365)^{25}$
1 mark	= 0.4313

d. What is the probability that *at least* two people have the same birthday? (1 mark)

1 mark	1-0.4313 = 0.568 (more than 57% likely!)

(b) Leo sits a multiple-choice exam. There are three options for a particular question (a, b and c). Leo has no idea of the answer or even what the question is asking, so randomly selects (a). Later in the exam, the teacher announces that (c) is not the answer. Leo thinks, excellent I now have 50% chance of selecting the right answer and decides there is no difference if he switches to (b) so remains with (a).

Is his reasoning correct in terms of probability? Explain by calculating the probability of being correct if he switches compared to the probability of being correct if he does not switch. (4 marks)

1 mark	Incorrect – Leo should switch.
1 mark	If you always plan to switch, only lose when the option initially selected was correct.
1 mark	The odds of selecting the right answer initially were 1/3 The odds of being wrong when Leo switches is 1/3
1 mark	If you switch, the chance of being correct is therefore 2/3 – DOUBLE the odds of not.

(c) Your car breaks down on the side of the road. You call someone who arrives and fixes the car by replacing the motor. In theory alone, is there a greater chance that the person who arrives is a lawyer or a lawyer and a mechanic? Explain. (3 marks)

1 mark	Lawyer
1 mark	Lawyer and mechanic are a subset of being a lawyer *This is a dependent event probability, like described at the start of the question)
1 mark	Requires P(lawyer) x P(mechanic), which will be less than P(lawyer)

(d) Three well behaved children have been nominated to receive a lollypop. There is, however, only one lollypop and the carer knows which child will receive it, but doesn't want to tell them yet. One of the children asks the carer "If Barry is to receive the lolly, say Charlotte's name instead. If Charlotte is to receive the lolly, say Barry's name. If I, Alice, will receive the lolly, flip a coin to decide which name

to say (either Barry or Charlotte).” The carer says that ***Barry*** will not be receiving the lolly.

Alice is pleased, thinking that her chances of receiving the sweet have increased from 1/3 to 1/2 as previously all three could receive the lolly, but now it’s just her and Charlotte in contention.

Alice tells Charlotte the news. Charlotte is also pleased. She claims that while Alice still has a 1/3 chance of receiving the lolly, she has a 2/3 chance of receiving the lolly.

Determine the probabilities of each child receiving the lolly after the carer’s statement to then conclude whether Alice or Charlotte are correct. (4 marks)

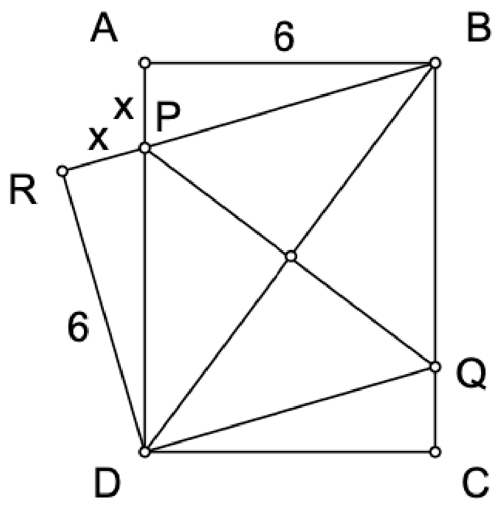
1 mark	Charlotte is correct
1 mark	Initially, each child had a 1/3 chance of receiving the lolly. As the carer said Barry will not receive the lolly, two options occur: 1. Charlotte will receive the lolly (1/3 chance); or
1 mark	2. Alice will receive the lolly and the coin flip appeared “Barry” (a 1/6 chance).
1 mark	Therefore, the chance that Alice receives the lolly are ½ that of Charlotte’s and Barry has no chance of receiving the lolly. Therefore, Alice’s chances of receiving the lolly are still 1/3 while Charlotte’s chance have doubled to 2/3.

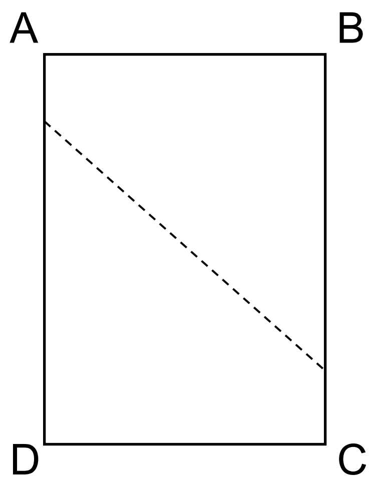
QUESTION 5: TABLECLOTH TRAINING

6 MARKS

You are given a rectangular tablecloth ABCD where AB = 6 cm and BC = 8 cm. The edges AB and BC have been folded over so that the corner B folds exactly onto the corner D. The tablecloth is then placed flat onto a table. What area of the table will now be covered?

Note: In a right angled triangle of sides *a*, *b* and *c*, $a^2 + b^2 = c^2$ and $area\ of\ a\ triangle = \frac{1}{2} \times a \times b$.





1 mark	*Area will be CQPRD* in the above image. This is the area of a triangle and trapezoid
1 mark	**Area of trapezoid** PDCQ = ½ of ABCD area = 6x8/2 = 24cm^2
1 mark	The length of PD = 8-x, therefore, $x^2 + 6^2 = (8-x)^2$ (**apply pythag**)
1 mark	So, x = 7/4 (**value of x**)
1 mark	Therefore, **area of PRD** = ½ x 6 x 7/4 = 21/4 = 5.25 cm^2
1 mark	In **total**, area = 24 + 21/4 = 29.25 cm^2 (1/2 mark off if NO UNITS)

QUESTION 6: WHAT'S THE CHANCE?

3 MARKS

Prove that if *x* = 0.999…… then x is also equal to 1 (0.9999 = 1) (hint: step one is to multiply by an integer!)

1 mark	Multiply by 10 so that 10 x = 9.999999
1 mark	Subtract an x 10-x = 9.9999 – 0.9999
1 mark	Therefore 9x = 9 so x =1

QUESTION 7: COMBINING CHANCES

10 MARKS

Combinations allow mathematicians to quickly determine the number of possible ways you can select *a* things from a total of *n* things. It is represented as *a*C*n*. For example, if I have 10 people and only 5 places on a team then you use 10C5 to determine the number of ways this can be done. To calculate 10C5, you can use your calculator (shift then divide sign for the 'C'), or the following formula (using the '!' button on your calculator):

$$aCn = \frac{n!}{r!\,(n-r)!}$$

(a) You have 20 chocolates, each a different flavour. You choose 5 and place these into one box. You then choose 5 of the remaining chocolates and place these into a second box and so on until you fill 4 boxes each with 5 chocolates. Complete the equation below to determine the number of ways the chocolates can be combined (3 marks):

	C		x		C		x		C		x		C		=	

20	C	5	x	15	C	5	x	10	C	5	x	5	C	5	=	11732745024

½ mark for each correct aCn and 1 mark for answer

(b) At a festival is a closed box containing 4 red marbles, 3 blue marbles and 6 green marbles. The storekeeper proclaims that to play his game one must remove three marbles without replacing them. You will win the game if all three marbles are the same colour.

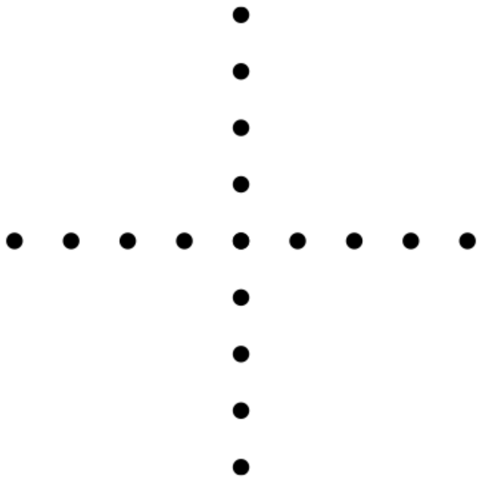
a. How many ways can you select three marbles of the same colour? (2 marks)

1 mark for all three separate multiplications	3C1x2C1x1C1 + 4C1x3C1x2C1 + 6C1x5C1x4C1
1 mark For the ADDITION of each multiplication (don't need to have 150 – accept carry through so long as they do an addition of separate events)	150 ways

b. What is the *chance* that you will select three marbles of the same colour and therefore win the game? (1 mark)

1 mark	150/(13C1x12C1x11C1) = 150/1716 = 8.74% chance (accept any decimals including 0).

(c) How many triangles (with an area greater than 0) are there with each of the three vertices (corners) at one of the dots in the diagram? (4 marks)



1 mark	Choose two points on the line across the page – 9C2 = 36 ways
1 mark	One point on the line vertically up the page (not the centre). This can be done in 8C1 = 8 ways.
1 mark	This gives a total of 8x36 = 288 triangles which can be obtained from two points on the line up and one across also so 576 triangles in total
1 mark	However, triangles with a vertex at the point where line meets have been counted twice – these are 8x8 = 64 of the options. So, in total 576-64 – **512 possible triangles.**
	Note an alternate route would be 17C3 (680) (17 dots and three needed for a triangle) (1 mark). Then, if all dots on the same line 9 area (1 mark) and this happens 2 x 9C3 ways = 168 (1 mark). In total then we have 680 – 168 = 512. (1 mark)

QUESTION 8: FIVE FEVER

4 MARKS

Consider the sequence 5, 55, 555, 5555, 55 555, ...

Is there any chance that any of the numbers in this sequence are divisible by 495? If so, what is the smallest number that could be?

1 mark	495 = 5 x 9 x 11 so a number is divisible by 495 IF divisible by all of 5, 9 and 11.
1 mark	As each number has a unit '5' ALL numbers are divisible by 5
1 mark	Every second term will be divisible by 11 but not the first/third/fifth (odd terms) as they have a remainder of 5 when divided by 11.
1 mark	The first even term divisible by 9 will be the solution. A number is divisible by 9 when its digit sum is divisible by 9. Therefore, 9 x 2 = 18 will be the first digit divisible by 495 = 555 555 555 555 555 555

QUESTION 9: SEQUENCE OF SEEDS

7 MARKS

Farmer Beatrice planted five lemon trees in a row. She then planted one apple tree in each of the spaces between the lemon trees. Next, she planted one rose plant in each of the spaces between the plants already planted. She repeated this process with tulips, lilies and finally daises.

(a) How many plants in total ended up in the row? (5 marks)

1 mark	After apples, 5 + 4 = 9 plants.
1 mark	Since there are now 9 plants, there are 8 spaces between plants. After rose plants, 9 + 8 = 17 plants.
1 mark	Since there are now 17 plants, there are 16 spaces between plants. After tulips, 17 + 16 = 33 plants.
1 mark	Since there are now 33 plants, there are 32 spaces between plants. After lilies, 33 + 32 = 65 plants.
1 mark	Finally, since there are now 65 plants, there are 64 spaces between plants. After daises, 65 + 64 = 129 plants. Therefore, a total of 129 plants in the row.

(b) If Farmer Beatrice was to plant 25 different types of plants using the same method, how many plants in total will end up in the row? (hint: derive a formula from the pattern you observed in (a)). (2 marks)

1 mark	Identify formula = $2^{n+1} + 1$
1 mark	For 25 plants, n = 25, there will be 67108865 plants