KNOX GRAMMAR SCHOOL

STATE

DA VINCI DECATHLON 2021

CELEBRATING THE ACADEMIC GIFTS OF STUDENTS

IN YEARS 9, 10 & 11

MATHEMATICS SOLUTIONS

TEAM NUMBER ________________

1	2	3	4	Total	Rank
/9	/16	/16	/9	/50	

QUESTION ONE

ROLL UP, ROLL UP

9 MARKS

When one thinks of **numbers** and **‘chance’**, a dice is the first thing that comes to mind. Almost everyone, it can be said with certainty, will picture a **six-sided** one. Dice with more or less sides **do exist** but are **very uncommon,** and at the most they might have a few extra sides – perhaps 12 at the maximum.

Dice Lab, however, a small Arizonan company, had other ideas, and set about creating a 120-sided dice back in 2016. The mathematical name for this shape is a **disdyakis triacontahedron**, and this is the focus of the present task!

1. How many edges does a disdyakis triacontahedron have? **180**

2. How many vertices does the shape have? **62**

3. What type of triangles are the faces? **Scalene**

4. Which feature of the shape does the equation below represent? **Volume**

$$\frac{180}{11}(5 + 4\sqrt{5})$$

5. Which feature of the shape does the equation below represent? **Surface Area**

$$\frac{180}{11}\sqrt{179 - 24\sqrt{5}}$$

6. Complete the equation, which represents the number of possible combinations of numbers from 1 to 120 that could be placed on the shape, by inserting a number in the brackets **(three marks)**.

$$N = 10^{(\mathbf{98})}$$

7. The above number is greater than the estimated number of atoms in the universe. True or false? **True**

QUESTION TWO

IN WITH A CHANCE

16 MARKS

Problems involving **probability** are among the most challenging in the mathematical world – but also the most **practical.** Can you answer those below?

Final answers are highlighted in yellow for each part.

QUESTION ONE (3 MARKS)

One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his assigned seat?

Look at the situation when the k'th passenger enters. Neither of the previous passengers showed any preference for the k'th seat vs. the seat of the first passenger. This in particular is true when k = n. But the n'th passenger can only occupy his seat or the first passenger's seat. Therefore, the probability is ½.

1 mark for final answer, 2 marks for reasoning.

QUESTION TWO (5 MARKS)

Mr. Smith works on the 13th floor of a 15 floor building. The only elevator moves continuously through floors 1, 2, . . . , 15, 14, . . . , 2, 1, 2, . . . , except that it stops on a floor on which the button has been pressed.

Assume that time spent loading and unloading passengers is very small compared to the travelling time. Mr. Smith complains that at 5pm, when he wants to go home, the elevator almost always goes up when it stops on his floor. What is the explanation?

Now assume that the building has n elevators, which move independently. Compute the proportion of time the first elevator on Mr. Smith's floor moves up.

In the one-elevator case, we can reasonably assume that the elevator is equally likely to be at any point between floor 1 and floor 15 at any point in time. We can also assume that the probability that the elevator is exactly on the 13th floor when Smith arrives is negligible. This gives the probability 2/14 = 1/7 ≈ 0.1429 that it is above floor 13 (which is when it will go down when it goes by this floor) when Smith wants to go home.

Let's have n elevators now. Call the unbiased portion the part of the elevator's route up from floor 9 to the top and then down to floor 13. Any elevator at a random spot of

the unbiased portion is equally likely to go up or down when it goes by the 13th floor. Moreover, if there is at least one elevator in the unbiased portion, all elevators out of it do not matter. However, if no elevator is in the unbiased portion, then the first one to reach the 13th floor goes up. Therefore, the probability that the first elevator to stop at 13th floor goes down equals ½ (1 − (10/14)n). (For n = 2 it equals approximately 0.2449.)

1 mark for each answer (2 total), 3 marks for reasoning.

QUESTION THREE (4 MARKS)

There are 64 teams who play single elimination tournament, hence 6 rounds, and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points.

If you have n round and 2n teams, the answer is ½ (2n − 1), so 31.5 when n = 6

1 mark for answer, 3 marks for reasoning (noting that the above is a simple, sophisticated method, but that others also exist)

QUESTION FOUR (4 MARKS)

You are a broker; your job is to accommodate your client's wishes without placing any of your personal capital at risk. Your client wishes to place an even $1,000 bet on the outcome of the World Series, which is a baseball contest decided in favour of whichever of two teams first wins 4 games. That is, the client deposits his $1,000 with you in advance of the series. At the end of the series he must receive from you either $2,000 if his team wins, or nothing if his team loses. No market exists for bets on the entire world 2 series.

However, you can place even bets, in any amount, on each game individually. What is your strategy for placing bets on the individual games in order to achieve the cumulative result demanded by your client?

Let's assume that the money unit is $1,000, call your team A and your client's team B. Call each sequence of games terminal if the series may end with it. To each terminal sequence at which A wins, say AAABA, attach value 2, and to each terminal sequence at which B wins, say BBAAABB, attach 0. These are of course the payoffs we need to make. Each non-terminal sequence, say AABA, will have a value which is the average of the two sequences to which it may be extended by the next game, AABAA and AABAB in this case. This recursively defines the values of all possible sequences. It is important to note that the value of the empty sequence (before games start) is 1, as the average on the sequences of length 7, and then at each shorter level, is 1. Now simply bet, on A, your value minus the lower value of your two successors at each sequence.

Note that you can extend, with 2's or respectively 0's, to length 7 all sequences in which A or respectively B wins. The value is the amount you have provided you use the above betting strategy. Also note that you do not need to split a penny because the values of sequences of length 1 have at most 25 in the denominator (and we know that the value is an integer for the sequence of length 0).

1 mark for answer, 3 marks for reasoning.

QUESTION THREE

LUCK OF THE DRAW

16 MARKS

Everyone dreams of winning the **lottery,** however, the odds are stacked against us. This question will reveal just how **unlikely** a lottery win actually is.

QUESTION ONE (4 MARKS)

A standard lottery sees 6 numbers drawn in a row, from a pool of balls numbered 1 to 49, without replacement. Calculate the chance of **winning** this lottery (i.e. of picking the 6 numbers in any order).

Answer (1 mark) – 1 in 13,983,816

Reasoning (3 marks) – there are multiple approaches; perhaps the most sophisticated is using the following formula:

$$\frac{49!}{6! * (49-6)!}$$

As explanation of the above formula:

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where n is the number of alternatives and k is the number of choices.

QUESTION TWO (4 MARKS)

Now, calculate the chance of picking just **one** correct number!

Answer (1 mark) – 1 in 2.4212, or 68,757/166,474

Reasoning (3 marks) – again, multiple approaches are possible, with the most sophisticated being the following formula:

$$\frac{\binom{6}{1}\binom{43}{5}}{\binom{49}{6}}$$

As explanation of the above formula:

This can be written in a general form for all lotteries as:

$$\frac{\binom{K}{B}\binom{N-K}{K-B}}{\binom{N}{K}}$$

where N is the number of balls in lottery, K is the number of balls in a single ticket, and B is the number of matching balls for a winning ticket.

QUESTION THREE (2 MARKS)

How about the chance of coming *oh so close* and picking **five** correct numbers?

Answer (1 mark) – 1 in 54,200.8, or 43/2,330,636

Reasoning (1 mark) – teams should be able to use the same process and/or formula employed for Q2

QUESTION FOUR (SIX MARKS)

Below are details for six types of lotteries across the world – order them from **most to least likely to be won.**

- **Irish Lotto** – 6 numbers, 47 balls
- **PowerBall** – 5 numbers plus 1 bonus number, 69 balls plus 26 bonus balls
- **MegaMillions** – 5 numbers plus 1 bonus number, 70 balls plus 25 bonus balls
- **EuroJackpot** – 5 numbers plus 2 bonus numbers, 50 balls plus 10 bonus balls
- **Oz Lotto** – 7 numbers, 45 balls
- **EuroMillions** - 5 numbers plus 2 bonus numbers, 50 balls plus 12 bonus balls

Answers:

1. **Irish Lotto** (most likely to be won)
2. **Oz Lotto**
3. **EuroJackpot**
4. **EuroMillions**
5. **PowerBall**
6. **MegaMillions** (least likely to be won)

One mark for every lottery in correct numbered position (i.e. one mark for EuroMillions being at #4 – marks are *not* to be awarded for correct relative order)

QUESTION FOUR

SIGNING OFF

9 MARKS

Probability involves a significant number of **mathematical signs and symbols**, many of which are rarely seen in any other fields. Are you able to **draw** those named in the table below (some, but not all of which, are probability-related)?

SIGN OR SYMBOL NAME	ANSWER (DRAWING)
Summation	$\sum$
'is proportional to'	$\propto$
Real numbers	**R**
'because'	$\because$
Population mean	μ
Standard deviation	σ
Combination	${}_nC_k$
Permutation	${}_nP_k$
	Note: for the above two answers, n and k may be x and y, or any other pronumerals ***Award half marks if only C and P are written as answers***
Delta	Δ