KNOX GRAMMAR SCHOOL

STATE

DA VINCI DECATHLON 2019

CELEBRATING THE ACADEMIC GIFTS OF STUDENTS IN YEARS 9, 10 & 11

da Vinci Decathlon® EXCITE ENRICH CHALLENGE An initiative of Knox Grammar School

MATHEMATICS SOLUTIONS

TEAM NUMBER _____

1	2	3	4	5	Total	Rank
/12	/8	/16	/12	/12	/60	

QUESTION ONE

BACKYARD MATHS

12 MARKS

Jim is doing some **landscaping** and gardening in his backyard, and he has the following **questions for you to complete**.

PART ONE (6 MARKS)

The edge of Jim's circular flowerbed, 220 feet in diameter, requires mulch. How many cubic yards of mulch is needed to cover 3 feet in from the edge, to a depth of 2.5 inches, all the way around the flowerbed (nearest yard)? There are 3 feet in a yard and 12 inches in a foot.

Total bed area = π x 110^2

Inside bed area = π x $(110-3)^2$

Therefore mulch area = (π x 110^2) – (π x $(110-3)^2$) **(1 mark)**

= 2045.176817 ft^2 **(1 mark)**

(teams may write this to 1/2/3… decimal points or nearest ft)

Now, note that 2.5 inches deep is 5/24 ft deep **(1 mark)**

So, volume of ring = 2045.176817 x 5/24

= 426.0785025 ft^3 **(1 mark)**

(teams may write "…")

Now, convert cubic feet to cubic yards – if there are 3 feet in a yard, then there must be 27 ft^3 in 1 yd^3 (3x3x3 = 27) - therefore 1 ft^3 is 1/27 yd^3 **(1 mark)**

Therefore the total cubic yards needed is 426.0785025 x 1/27

= 15.78…

= 16 yd^3 (nearest yard) **(1 mark)**

(Total 6 marks)

PART TWO (3 MARKS)

Jim has 5 very large pots that take 1.75 bushels of potting mix each. A bag of potting mix contains about 3 cubic feet. How many bags are needed? One bushel is 5 April in cubic feet.

1.75 bushels x 5 pots = 8.75 bushels (1 mark)

5 April = 5/4 = 1.25 (1 mark)

(yes, this is an intentionally cryptic question)

8.75 x 1.25 = 11 ft^3

11/3 = 3.666666

Therefore 4 bags will be needed (1 mark)

(Total 3 marks)

PART THREE (3 MARKS)

Finally, Jim has a kidney-shaped garden bed and wants to plant some herbs in it. Each plant will be 1ft away from each other. The shape is irregular and the width varies, but Jim nevertheless measured the width at intervals of 10ft, starting from one end of the bed. The widths recorded were 11ft, 13ft, 5ft and 17ft. How many plants will Jim need to fill the bed?

Teams should note that basically, several rectangles are being added together – although the edges are curved, we are not looking for a precise area and so it is OK to proceed in this manner

Therefore, we have an approximate area of 11x10 + 13x10 + 5x10 + 17x10

= 460ft^2 (1 mark)

As the plants are to be 1ft apart, it should then be noted that there is one square foot per plant (1 mark)

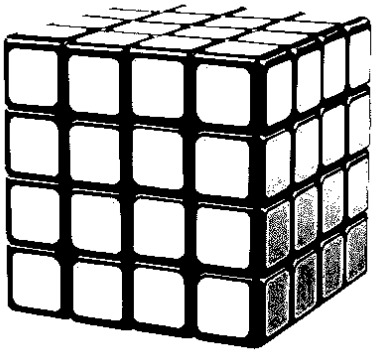
That means 460 plants in total (1 mark)

(Total 3 marks)

QUESTION TWO

CHANGING THE GAME

8 MARKS



There is a **new product** on the market, which is set to change the gaming landscape. Gone are the days of regular tic-tac-toe, for a much greater invention has come to light – **three-dimensional** noughts and crosses.

The game is played using a 4x4x4 board (see right), composed of 64 cells, and players take turns as usual. The first player to get **four in a row** of their symbol wins. Players place counters inside the cells – imagine that the board is comprised of 64 see-through boxes.

How many different ways can the game be won? Furthermore, what would be the formula for the number of ways to win on an n-dimensional board of k height/width/depth?

Note that in a 6x6x6 board, there is a 4x4x4 inner cube (1 mark)

Each winning line of the 4x4x4 inner cube intersects 2 distinct 1x1x1 cubes in the outer shell, and each 1x1x1 cube is the continuation of one winning line (1 mark)

Therefore the number of ways to win is ½ the number of outer 1x1x1 cubes (1 mark)

This is equal to ½ x (6^3-4^3) (1 mark)

Therefore there are 76 distinct ways to win (2 marks)

(NOTE: there are other ways to reach this answer, but the above is the most sophisticated – teams who get the correct answer should be awarded 2 marks for that, plus a discretionary number of extra marks for the degree of sophistication of their working – to a maximum of 4, but again noting that the above method is the most sophisticated option and it would be difficult to match it through other methods such as trial and error)

The formula required for the second part of this question is:

$$\frac{1}{2}[(k+2)^2 - k^n]$$

(2 marks for this formula)

(Total 8 marks)

QUESTION THREE

SHIP SHAPE

16 MARKS

Two ships, S and T, are cruising on **straight courses** and at **constant speeds**. At 10.00am, they are 5km apart, at 11.00am they are 4km apart. By 1.00pm they are 10km apart. At 7.00am, S was due west of T.

Using this information, please answer the **following questions**:

a) How far apart were the ships at 7.00am? **(4 marks)**

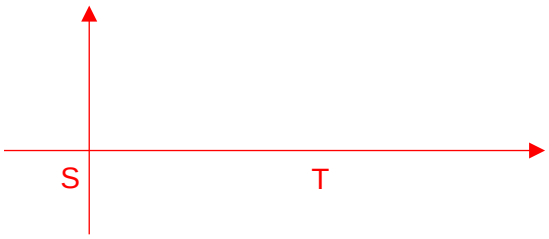
b) When in future will they be 26km apart? **(2 marks)**

c) How close do the ships get to each other? At what time are they nearest? **(2 marks)**

d) When is S due north of T? **(2 marks)**

e) When is S southwest of T? **(3 marks)**

f) Suppose S and T have the same speed and T is heading due south. What is the speed of S? **(3 marks)**



(a)

The above diagram represents the starting point of S and T at 7.00am.

Let the initial coordinates of S be (a, b) and the distance ST (at 7.00am) be m. Let the initial position of T be (c, d).

The coordinates of S and T after *t* hours are therefore (at, bt) and (m + ct, dt) respectively.

Now use the distance formula for each of the distances apart that we are given

$(m+3c-3a)^2 + (3d-3b)^2 = 5^2$

$(m+4c-4a)^2 + (4d-4b)^2 = 4^2$

$(m+6c-6a)^2 + (6d-6b)^2 = 10^2$ **(1 mark for these)**

Let u = c-a, v = d-b, $u^2 + v^2 = x^2$

Now, after factorising, we will have:

$m^2 + 6mu + 9x^x = 25$

$m^2 + 8mu + 16x^x = 16$

$m^2 + 12mu + 36x^x = 100$ **(1 mark for these)**

(i.e. teams who do not simplify will not receive this mark, but can still get the final answer)

By subtraction (simultaneous equations)

$2mu + 7x^2 = -9$

$4mu + 20x^2 = 84$ **(1 mark for these)**

(or 1 mark for equivalent, if teams choose not to simplify by using u, v and x)

Solving, $x^2 = 17$ and $mu = -64$

Substituting back into one of the initial equations produces $m^2 = 256$

Therefore m = 16km **(1 mark for this answer)**

(b)

Using distance formula again,

$m^2 + 2tmu + t^2x^2 = 26^2$ **(1 mark)**

Therefore, $256 – 128t + 17t^2 = 26^2$ (by substituting in 17 and -64 from a))

This simplifies down to $(17t + 42)(t – 10) = 0$

t = 10 and -42/17

Therefore the answer for the FUTURE time is 10 hours after 7.00am

i.e. 5pm **(1 mark)**

(c)

Let us take the distance formula again (from b))

$D(t)^2 = 256 – 128t + 17t^2$

If we complete squares, $D(t)^2 = 17(t – 64/17) + 256/17$

The closest distance will thus be when the $17(t – 64/17) = 0$

$D(t)^2$ thus = 256/17, and the square root of this, which will be the shortest distance between the ships, is 3.88km (2 d.p) **(1 mark)**

This occurs when t = 64/17

Therefore the time that it occurs at is 7.00am + 64/17 hours

This is at 10.46 am (nearest minute) (1 mark)

(d)

For this part, the x-values of S and T will be the same (due north) (1 mark)

Thus at = m + ct

We know m = 16 and c-a = u = -4

Substituting and solving gives t = 4

Therefore this occurs at 7.00am + 4 = 11.00am (1 mark)

(e)

This will be when $T_x - S_x = T_y - S_y > 0$ (1 mark)

Therefore m + ut = vt > 0

m = 16, u = -4 as we know, and from a), $u^2 + v^2 = x^2$ ($x^2 = 17$), meaning that $v^2 = 17 - (-4)^2$ and thus v = ±1

Therefore we have:

16 – 4t = ±1t (1 mark)

v = -1 is not possible, and so we use v = 1 only

This gives t = 16/5

The time is therefore 10.12am (1 mark)

(f)

For these conditions, c = 0 and $a^2 + b^2 = c^2 + d^2$ (1 mark)

c-a = u = -4, and so a = 4

Meanwhile $(d-b)^2 + a^2 = 17$ and so d-b = ±1

Solving $d^2-b^2 = 16$ with d-b = ±1 gives d = -17/2 and b = -15/2 (1 mark)

S is therefore moving at $\sqrt{4^2 + (\frac{15}{2})^2}$ = 17/2 kph (1 mark)

(Total 16 marks)

QUESTION FOUR

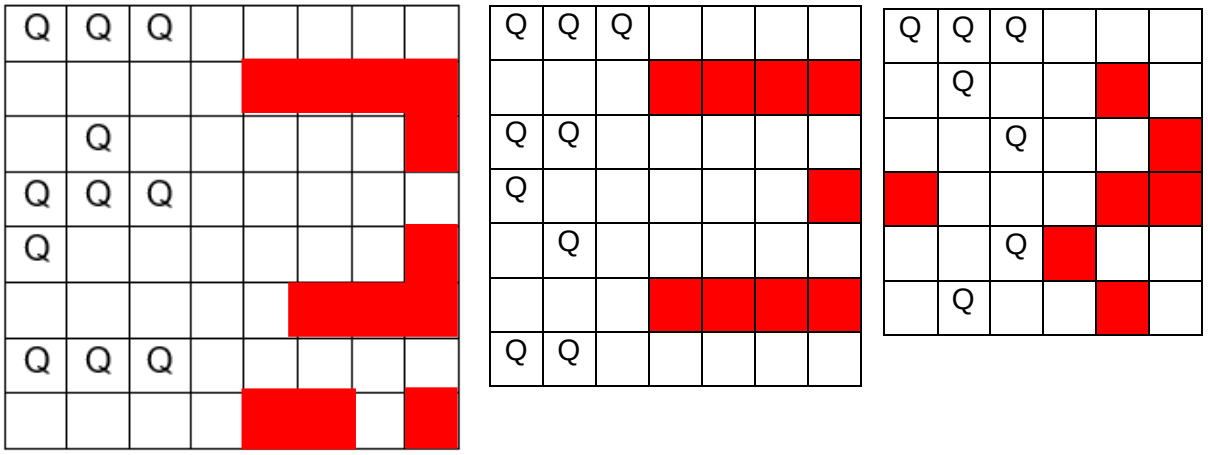
ACROSS THE BOARD

12 MARKS

The **chessboard** is one of the ultimate **mathematical landscapes**, used for centuries as a form of leisure, entertainment, competition and challenge. Below are a number of chess-based puzzles for you to complete.

PART ONE – SINGLE QUEENS (6 MARKS)

This task is simple. Shade all the squares controlled by a single queen only.



Marking guide – each square is out of 2 marks. ½ mark off for each cell that is either incorrectly shaded or incorrectly not shaded (i.e. 4 or more mistakes = 0/2)

PART TWO – LATIN SQUARE (3 MARKS)

This problem is, in reality, more like sudoku than chess. Can you complete the square?

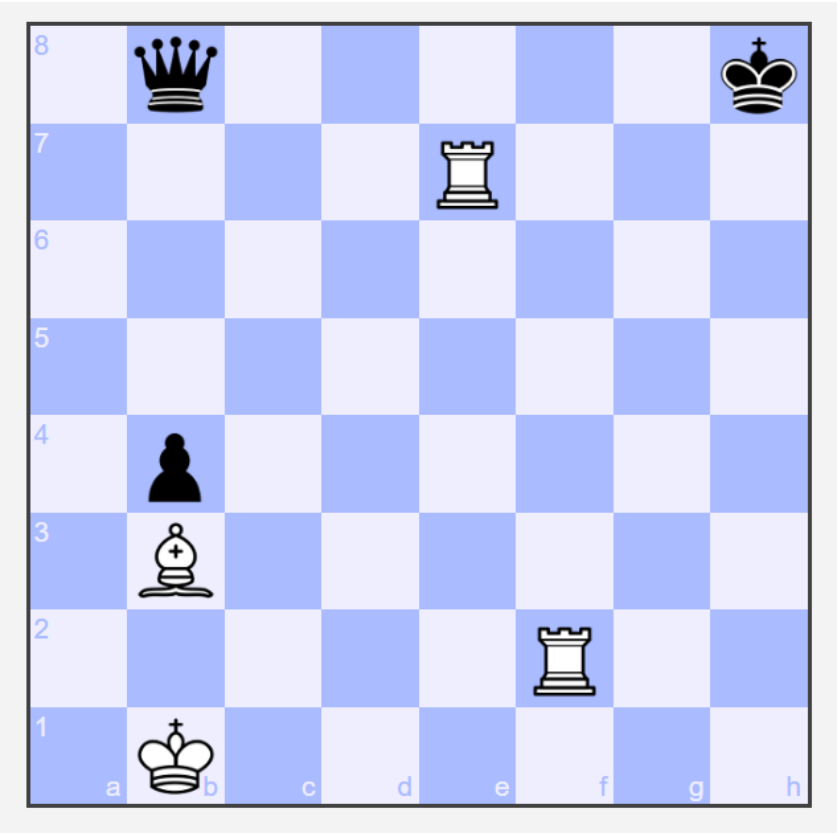
D	**F**	A	**E**	**C**	**B**
B	**D**	**C**	**F**	**A**	E
A	**B**	**E**	**D**	**F**	**C**
F	**E**	B	C	D	**A**
E	**C**	**D**	**A**	**B**	F
C	**A**	**F**	**B**	**E**	**D**

Marking guide – all correct – 3/3

Each mistake – ½ mark off (i.e. 6 or more mistakes = 0/3)

PART THREE – CHECKMATE (3 MARKS)

Finally, a traditional puzzle – can you checkmate in three moves? Please provide your answer in the space underneath. You are white, and it is your move.



Move one – Rook (F2) to A2 **(1 mark)**

Move two – Rook (A2) to H2 **(1 mark)**

Move three – Rook (H2) to H5 (taking Queen, and checkmate) **(1 mark)**

Note: there are a select few moves that black could make in between these moves. This does not affect the outcome – the three moves listed above are possible in any case and are the only correct answer.

QUESTION FIVE

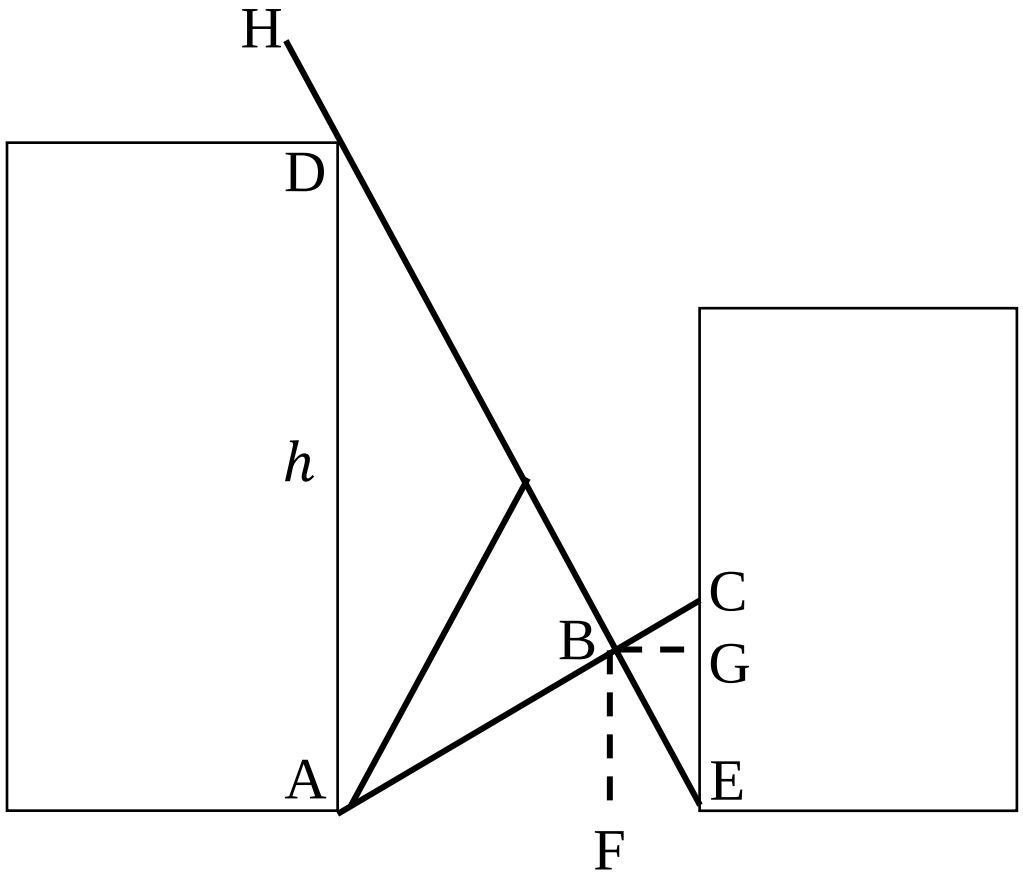
LADDER LINKUP

12 MARKS

This problem is represented by the **diagram below**. In essence, there are **two buildings** on opposite sides of a street, and a **series of ladders** laid across the space in between. Your task is to **determine h,** the height of the **taller building** (i.e. length DA).

Length HD is 2 feet 2 inches, while ladder AC is 22 feet 11 inches. Ladder HE is 70 feet and length BF is 5 feet 10 inches.

For reference, one foot is twelve inches. The capital letters in the diagram represent **points**. Please use them in your working out.



Hint: You will eventually reach the number 586,971 as one side of an equation. This can be expressed as:

1. $1386^2 - 1155^2$
2. $814^2 - 275^2$
3. $770^2 - 77^2$

ADE and ACE are right angled triangles

Thus $DE^2 = DA^2 + AE^2$ and $AC^2 = CE^2 + AE^2$ **(1 mark)**

Subtracting one from the other gives $DE^2 - AC^2 = DA^2 - CE^2$ **(1 mark)**

Meanwhile triangles ABD and CBE are similar

The following proportion thus exists:

DA : CE = AB : BC **(1 mark)**

The same can be said of ABF and BCG

Therefore AB : BC = BF : CG **(1 mark)**

Combining these proportions gives:

DA : CE = BF : CG, and thus DA = (CE x BF) / CG **(1 mark)**

Substitute this in for DA from above…

$DE^2 - AC^2 = ((CE \times BF) / CG)^2 - CE^2$ **(1 mark)**

Similarly, substitute CE for CG + BF, giving:

$DE^2 - AC^2 = (((CG+BF) \times BF) / CG)^2 - CE^2$ **(1 mark)**

We have the actual values for many of these terms. Let us now substitute them (all in inches only – converted from feet and inches), and replace CG with x for simplicity.

$814^2 - 275^2 = (70^2/x + 70)^2 - (x+70)^2$ **(1 mark)**

(NOTE that DE = 814 because the full length of the ladder is 70ft which is 840 inches, but the 2 feet 2 inches overhang (i.e. HD) needs to be removed from this for the triangle side length)

The value on the left is 586,971, which is where the hint comes in to play. The third difference of squares is very similar to the equation we have, and if we equate terms and remove squares we have:

$(70^2/x + 70) = 770$ and $(x+70) = 77$ **(1 mark)**

This means x = 7 = CG, while CE = CG + BF (BF is known)

(so, CG = 7, BF = 5 ft 10 = 70, CE = 77)

Substitute this back into the equation for DA again:

DA = (77 x 70)/7 = 770 inches = 64 feet 2 inches (3 marks)

(total 12 marks)

(NOTE: as usual there are different ways to reach this answer – the answer itself is worth 3 marks, and the working is worth 9 – use discretion as to the correctness and sophistication of any working that is different to the method above)